Wave Propagation in a Nonparabolic Graded-Index Medium

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***Abstract*—Wave propagation in a slab of graded-index medium whose refractive index is a polynomial expression of the transverse coordinate is treated by a simple generalization of the theory commonly adopted for the square-law media.**

WAVE PROPAGATION in graded-index media has attracted much attention in recent years, especially as regards the square-law media, whose refractive index n is given by

$$n^2(x)=n_0^2-n_2x^2. \tag{1}$$

For such media, exact solutions of the (scalar) wave equation exist [1] and are expressed by Hermite–Gaussian functions, namely by

$$u_m(x)=H_m\left(\frac{\sqrt{2}\,x}{w}\right)\exp\left(-\frac{x^2}{w^2}+i\beta_m z\right) \tag{2}$$

where

$$w^2=\frac{2}{k\sqrt{n_2}}$$
$$\beta_m^2=k^2n_0^2-k(2m+1)\sqrt{n_2} \tag{3}$$

and H_m denotes the Hermite polynomial of order m.

However, (1) is usually only the paraxial approximation of the square of the refractive index of a medium, which clearly is more accurate for narrower beams. Such an approximation can be improved by assuming

$$n^2(x)=n_0^2-n_2x^2+n_4x^4-n_6x^6+\cdots=P_l(x) \tag{4}$$

where P_l indicates an (even) polynomial of degree l larger than 2. The propagation in media specified by a polynomial expression of n^2 of type (4) has been studied by several authors [2], [3], and [4] with different methods and different approximations. The wave-optics procedure described here appears to be easier and yields results that are formally simpler than the previous ones. By limiting ourselves, for simplicity, to the TE case, our approximation consists in looking for solutions of the wave equation of the type

$$u_m(x)=K_m\left(\sqrt{2}\,\frac{x}{w}\right)\exp\left(ik\Omega_j(x)+i\beta z\right) \tag{5}$$

where K_m and Ω_j are polynomials of degree m and j, respectively. For simplicity, we consider here only the cases where $\Omega_j(x)$ is an even polynomial of x, so that K_m turns out to be either even or odd, depending on the parity of m. We will assume m to be even, too; however, the results are completely analogous when m is odd.

The introduction of (5) into the wave equation yields a polynomial equation in x, namely,

$$\frac{2}{w^2}K_m''+2\sqrt{2}\,\frac{ik}{w}\Omega_j'K_m'+ik\Omega_j''K_m+\left[k^2\left(P_l-\Omega_j'^2\right)-\beta^2\right]K_m=0 \tag{6}$$

where the primes indicate derivatives with respect to the argument.

If we choose $l=2j-2$, as indicated by the last term on the left-hand side of (8), which is the highest order term, (6) gives rise to $m/2+j$ equations (obtained by letting the coefficient of each power of x vanish), while the parameters to be determined are $m/2+j/2+1$, namely, the $m/2$ coefficients of K_m, which is to be determined apart from a constant factor, the $j/2$ coefficients of Ω_j, which is to be determined apart from an additive constant, and β. The number of equations, therefore, is equal to the number of unknowns, only if $j=2$ (and, therefore, $l=2$, which corresponds to a quadratic medium). In any other case, the number of equations is larger than the number of unknowns. Consequently, for $l>2$, the wave equation cannot be solved exactly but only approximately. The approximation consists in considering not all equations derivable from (6), but only the first $m/2+l/2+1$ equations, involving all powers of x up to x^{m+l}. In the present note we consider the case $l=j=4$, but the procedure can be extended to any order.

We will put

$$\Omega_4(x)=\frac{x^2}{2q}+\frac{x^4}{4Q^3}. \tag{7}$$

The treatment is simplified by assuming K_m to satisfy a differential equation of the type

$$K_m''(X)-2Xp(X)K_m'+V(X)K_m(X)=0 \tag{8}$$

where X indicates the argument $\sqrt{2}\,x/w$, and $p(X)$, $V(X)$ denote two polynomials to be determined. The introduction of (8) into (6) yields

$$ik\Omega_4'+\frac{2x}{w^2}p\left(\sqrt{2}\,\frac{x}{w}\right)=0 \tag{9}$$

Manuscript received November 7, 1977; revised March 28, 1978.
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0018-9480/78/1100-0856$00.75

$$-\frac{2}{w^2}V\left(\sqrt{2}\,\frac{x}{w}\right)+ik\Omega_4''+k^2(P_4-\Omega_4'^2)-\beta^2=0. \quad (10)$$

Equation (10) may be satisfied if $p(X)$ is a second-order polynomial. Since $V(X)$ has the same degree as $p(X)$, as follows from (8), we will write

$$p(X)=1+p_2X^2$$
$$V(X)=2m(1+V_0)+V_2X^2. \quad (11)$$

The coefficients p_2, V_0, and V_4 are not all independent of one another, since the condition has to be satisfied that K_m is a polynomial. Such coefficients are functions of n_4 and tend to zero when n_4 tends to zero, since K_m tends to the Hermite polynomial of order m, for which $p(X)=1$ and $V(X)=2m$. In the limit of very small n_4, the following relations may be verified (see (A5) and (A7) of the Appendix):

$$V_2=2mp_2$$
$$V_0=\frac{1}{2}(m-1)p_2. \quad (12)$$

From (9) it follows

$$\frac{ik}{2q}=-\frac{1}{w^2}$$
$$\frac{ik}{4Q^3}=-\frac{1}{w^4}p_2. \quad (13)$$

Then, (10), by neglecting all powers of x larger than x^4, yields

$$\beta^2=k^2n_0^2+\frac{ik}{q}-\frac{4m}{w^2}(1+V_0) \quad (14)$$

and

$$\frac{k^2}{q^2}+\frac{4}{w^4}V_2-3\frac{ik}{Q^3}=-k^2n_2$$
$$\frac{2}{qQ^3}=n_4. \quad (15)$$

Up to the first order in n_4, (15) with the help of (13) and (12) yields

$$p_2=-\frac{1}{2k}\frac{n_4}{n_2^{3/2}} \quad (16)$$

and

$$\frac{1}{w^2}=\frac{1}{2}k\sqrt{n_2}\left[1-\frac{2m+3}{4k}\frac{n_4}{n_2^{3/2}}\right]$$
$$\frac{ik}{4Q^3}=\frac{1}{8}k\frac{n_4}{\sqrt{n_2}}. \quad (17)$$

Then (14) yields

$$\beta^2=k^2n_0^2-k(2m+1)\sqrt{n_2}+\frac{3}{4}\frac{n_4}{n_2}(2m^2+2m+1) \quad (18)$$

which coincides with the expressions given in [3] and [4] and applied in [5].

In (17) and (18) terms of the order of $1/ik$ have been neglected compared with terms of the order of 1. It can also be noted that the above formulas hold for a possibly complex refractive index. When $n(x)$ is complex, the sign of $\sqrt{n_2}$ is to be so chosen as to satisfy the radiation conditions at infinity, for $|x|\to\infty$.

By solving (8) in a first approximation with respect to n_4, one finds (see Appendix)

$$K_m=\Sigma c_nX^n\left[1-\frac{1}{8}p_2(m-n)(3m+n-4)\right]$$

where c_n denotes the coefficient of X^n in the Hermite polynomial of order m and argument X; hence

$$K_m=\left[1-\frac{m}{8}(3m-4)p_2\right]H_m(X)+\frac{1}{8}(2m-3)$$
$$\cdot p_2XH_m'(X)+\frac{1}{8}p_2X^2H_7''(X)$$
$$\cong\left(1-\frac{m}{4}p_2X^2\right)H_m(X)+\frac{1}{8}p_2(2m-3+2X^2)XH_m'(X). \quad (19)$$

In conclusion, a graded-index medium specified by

$$n^2=n_0^2-n_2x^2+n_4x^4$$

sustains beams of the type

$$u_m(x,z)=K_m\left(\sqrt{2}\,\frac{x}{w}\right)\exp\left[i\beta z-\frac{x^2}{w^2}-p_2\frac{x^4}{w^4}\right]$$

where $1/w^2$ is given by the first equation in (17), p_2 by (16), β by (18), and K_m satisfies (8). In a first approximation with respect to n_4, K_m is given by (19), with $X=\sqrt{2}\,x/w$.

Appendix

Let us put

$$K_m(X)=\Sigma a_nX^n \quad (A1)$$

with

$$a_n=c_n(1+\Delta_n) \quad (A2)$$

and c_n denotes the nth coefficient of the Hermite polynomial of order m, namely,

$$c_n=\frac{(-1)^{(1/2)n}2^n}{n!\left(\frac{m}{2}-\frac{n}{2}\right)!}\left(\frac{m}{2}\right)!, \qquad \text{for } n \text{ even}, \qquad 0\leqslant n\leqslant m$$

$$c_n=0, \qquad \text{for } n \text{ odd}, \qquad n<0, \qquad n>m. \quad (A3)$$

The introduction of (A1) into (8), where $p(X)$ and $V(X)$ are given by (11), yields

$$\Sigma n(n-1)a_nX^{n-2}-2(1+p_2X^2)\Sigma na_nX^n$$
$$+\left[2m(1+V_0)+V_2X^2\right]\Sigma a_nX^n=0 \quad (A4)$$

hence equating to zero the coefficient of X^{m+2}

$$V_2=2mp_2. \quad (A5)$$

Moreover, (A4) yields the recurrent formula

$$(n+2)(n+1)a_{n+2}+2\left[(m-n)+mV_0\right]a_n$$
$$+2(m-n+2)p_2a_{n-2}=0. \quad (A6)$$

The condition $a_{m+2}=0$ can be written as

$$mV_0a_m+2p_2a_{m-2}=0$$

and, therefore,

$$V_0=\frac{1}{2}(m-1)p_2 \qquad \text{(A7)}$$

where, for small V_0 and p_2, we have used the expression $a_n=c_n$. Then, by using (A2), (A3), and (A7), (A6) can be written in the form

$$\Delta_{n+2}-\Delta_n=\frac{1}{2}(m+n-1)p_2, \qquad n \text{ even} \qquad \text{(A8)}$$

which yields

$$\Delta_n=\frac{1}{8}n(2m+n-4)p_2+\Delta_0. \qquad \text{(A9)}$$

By choosing Δ_0 so that $\Delta_m=0$, (A9) can be rewritten as

$$\Delta_n=-\frac{1}{8}(m-n)(3m+n-4)p_2. \qquad \text{(A10)}$$

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The Screening Potential Theory of Excess Conduction Loss at Millimeter and Submillimeter Wavelengths

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Abstract—**On using the screening potential theory, the room-temperature excess conduction loss in copper waveguide is explained and calculated. The low-frequency and long-wavelength conductivity with spatial dispersion has been shown to give 30-percent more conduction loss in copper at submillimeter-wave frequency. Good agreement between experimental and theoretical results is obtained.**

Recent measurements of the surface resistance of single-crystal copper by Tischer indicate that a room-temperature anomalous skin effect exists at millimeter-wave and upper microwave frequencies [1]–[3]. When extrinsic effects, i.e., surface roughness, waveguide size deviation, temperature, corrosion, work hardening, and oxygen absorption are taken into account and subsequently excluded, there is observed an anomalous skin effect which gives a 13.5-percent higher measured surface resistance than the classical theory can account for at 35 GHz and room temperature. It increases to 20 percent higher at 70 GHz. It is also reported for gold [5].

Manuscript received November 10, 1977; revised March 20, 1978. This work was supported by the NASA Goddard Space Flight Center.

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It is easily seen that the anomaly cannot be explained by either the Drude model or Pippard's anomalous skin effect. The anomaly can be instead attributed to the spatial dispersion of the conductivity $\sigma(q, \omega)$ due to the charge-density fluctuation induced screening potential [4]. The conductivity in the MKS units can be calculated from Harrison's dielectric function [4] (see Appendix) as given by

$$\sigma(q,\omega)=\sigma_r+i\sigma_i=-3i\omega\tau\sigma_0K/(qv\tau)^2 \qquad (1)$$

$$K=\frac{1-\dfrac{1-i\omega\tau}{2iqv\tau}\ln\left(\dfrac{1-i\omega\tau+iqv\tau}{1-i\omega\tau-iqv\tau}\right)}{1-\dfrac{1}{2iqv\tau}\ln\left(\dfrac{1-i\omega\tau+iqv\tau}{1-i\omega\tau-iqv\tau}\right)} \qquad (2)$$

where for copper $\tau=2.37\times10^{-14}$s, $v=v_F=1.58\times10^6$ m/s, $\sigma_0=e^2n_0\,\tau/\text{m}=\text{dc conductivity}=5.80\times10^7$ S/m; all field variables vary according to $\exp i\,(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t)$. In deriving (1) and (2) it is assumed that the currents and fields will have the same dependence on position, which is reasonable because of the small mean-free path at low frequency and room temperature; consequently, electrons moving at all

0018-9480/78/1100-0858$00.75